Chapter 23: Electrostatic Energy and Capacitance

Capacitors and Capacitance

- Capacitor

  - Any two conductors separated by an insulator (or a vacuum) form a capacitor.
  - In practice each conductor initially has zero net charge and electrons are transferred from one conductor to the other (charging the conductor).
  - Then two conductors have charge with equal magnitude and opposite sign, although the net charge is still zero.
  - When a capacitor has or stores charge $Q$, the conductor with the higher potential has charge $+Q$ and the other $-Q$ if $Q>0$. 
Capacitors and Capacitance

- Capacitance

- One way to charge a capacitor is to connect these conductors to opposite terminals of a battery, which gives a fixed potential difference \( V_{ab} \) between conductors (a-side for positive charge and b-side for negative charge). Then once the charge \( Q \) and \(-Q\) are established, the battery is disconnected.

- If the magnitude of the charge \( Q \) is doubled, the electric field becomes twice stronger and \( V_{ab} \) is twice larger.

- Then the ratio \( Q/V_{ab} \) is still constant and it is called the capacitance \( C \).

\[
C = \frac{Q}{V_{ab}} \quad \text{units} \quad 1 \text{ F} = 1 \text{ farad} = 1 \text{ C/V} = 1 \text{ coulomb/volt}
\]

- When a capacitor has or stores charge \( Q \), the conductor with the higher potential has charge \(+Q\) and the other \(-Q\) if \( Q>0 \)
Calculating Capacitance

- Parallel-plate capacitor in vacuum

- Charge density: \( \sigma = \frac{Q}{A} \)

- Electric field: \( E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \)

- Potential difference: \( V_{ab} = Ed = \frac{1}{\varepsilon_0} \frac{Qd}{A} \)

- Capacitance: \( C = \frac{Q}{V_{ab}} = \frac{\varepsilon_0}{d} \frac{A}{A} \)

- The capacitance depends only on the geometry of the capacitor.
- It is proportional to the area A.
- It is inversely proportional to the separation d.
- When matter is present between the plates, its properties affect the capacitance.
Calculating Capacitance

- **Units**

  \[1 \text{ F} = 1 \frac{C^2}{N \text{ m}} \text{ (Note } \varepsilon_0 = \frac{C^2}{N \text{ m}^2}) \]

  \[1 \mu \text{ F} = 10^{-6} \text{ F}, \ 1 \text{ pF} = 10^{-12} \text{ F}\]

  \[\varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m}\]

- **Example 24.1: Size of a 1-F capacitor**

  \[d = 1 \text{ mm}, \ C = 1.0 \text{ F}\]

  \[A = \frac{Cd}{\varepsilon_0} = \frac{(1.0 \text{ F})(1.0 \times 10^{-3} \text{ m})}{8.85 \times 10^{-12} \text{ F/m}} = 1.1 \times 10^8 \text{ m}^2\]
Calculating Capacitance

Example 24.2: Properties of a parallel capacitor

A parallel-palte capacitor in vacuum

d = 5.00 mm, \( A = 2.00 \text{ m}^2 \), \( V = 10,000 \text{ V} = 10.0 \text{ kV} \)

\[
C = \varepsilon_0 \frac{A}{d} = \frac{(8.85 \times 10^{-12} \text{ F/m})(2.00 \text{ m}^2)}{5.00 \times 10^{-3} \text{ m}}
\]

= \(3.54 \times 10^{-5} \text{ F} = 0.00354 \mu\text{F}\)

\[
Q = CV_{ab} = (3.54 \times 10^{-9} \text{ C/V})(1.00 \times 10^4 \text{ V})
\]

= \(3.54 \times 10^{-5} \text{ C} = 35.4 \mu\text{C}\)

\[
E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} = \frac{3.54 \times 10^{-5} \text{ C}}{(8.85 \times 10^{-12} \text{ C}^2 / \text{N} \cdot \text{m}^2)(2.00 \text{ m}^2)}
\]

= \(2.00 \times 10^6 \text{ N/C}\)
Calculating Capacitance

Example 24.3: A spherical capacitor

From Gauss’s law:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0}$$

$\vec{E}$ is constant in magnitude and parallel to $d\vec{A}$
at every point on a sphere as a Gaussian surface

$$E(4\pi r^2) = \frac{Q}{\varepsilon_0} \rightarrow E = \frac{Q}{4\pi \varepsilon_0 r^2}$$

This form is the same as that for a point charge

$$V = \frac{Q}{4\pi \varepsilon_0 r}$$

$$V_{ab} = V_a - V_b = \frac{Q}{4\pi \varepsilon_0 r_a} - \frac{Q}{4\pi \varepsilon_0 r_b} = \frac{Q}{4\pi \varepsilon_0} \frac{r_b - r_a}{r_a r_b}$$

$$C = \frac{Q}{V_{ab}} = 4\pi \varepsilon_0 \frac{r_a r_b}{r_b - r_a}$$
Example 24.4: A cylindrical capacitor (length L)

\[ V = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r_0}{r} \text{ from Example 23.10} \]

\[ C = \frac{Q}{V_{ab}} = \frac{\lambda L}{2\pi\varepsilon_0} \ln \frac{r_b}{r_a} = \frac{2\pi\varepsilon_0 L}{\ln \frac{r_b}{r_a}} \]
Capacitors in Series and Parallel

- Capacitors in series

Some symbols:

- Capacitor
- Battery
- Conducting

Diagram:

- Capacitor
- Battery
- Conducting

Symbol:

- Capacitor (C)
- Voltage (V)
The equivalent capacitance $C_{eq}$ of the series combination is defined as the capacitance of a single capacitor for which the charge $Q$ is the same as for the combination, when the potential difference $V$ is the same.

$$C_{eq} = \frac{Q}{V} \quad \frac{1}{C_{eq}} = \frac{V}{Q} \quad \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \quad \frac{1}{C_{eq}} = \sum_{i} \frac{1}{C_i}$$
Capacitors in Series and Parallel

- Capacitors in parallel

\[ Q_1 = C_1 V \quad Q_2 = C_2 V \]
\[ Q = Q_1 + Q_2 = (C_1 + C_2)V \]

\[ \frac{Q}{V} = C_1 + C_2 \]

The parallel combination is equivalent to a single capacitor with the same total charge \( Q = Q_1 + Q_2 \) and potential difference.

\[ C_{eq} = C_1 + C_2 \Rightarrow C_{eq} = \sum_i C_i \]
Capacitors in Series and Parallel

- Capacitor networks

Example 1: Find the equivalent capacitance between $a$ and $b$ for the combination of capacitors shown. All capacitances are in microfarads.

\[
\frac{1}{C_{eq}} = \frac{1}{12} + \frac{1}{6} = \frac{3}{12} \Rightarrow C_{eq} = 4
\]
Capacitors in Series and Parallel

- Capacitor networks (cont’d)

\[ C_{eq} = 4 + 11 + 3 = 18 \]

\[ \frac{1}{C_{eq}} = \frac{1}{18} + \frac{1}{9} = \frac{3}{18} \Rightarrow C_{eq} = 6 \]

\[ C_{eq} = 6.0 \mu F \]
Capacitors in Series and Parallel

- Capacitor networks 2

\[
\begin{array}{c}
\text{A} \\
C \quad C \\
\text{B} \\
C \quad C \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{A} \\
1 \quad C \\
\text{B} \\
1 \quad C \\
\end{array}
\]

\[
\begin{array}{c}
\text{A} \\
\frac{4}{3} \quad C \\
\text{B} \\
C \quad C \\
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{A} \\
\frac{15}{41} \quad C \\
\text{B} \\
\end{array}
\]
Energy Storage and Electric-field Energy

- Work done to charge a capacitor

  - Consider a process to charge a capacitor up to \( Q \) with the final potential difference \( V \).
    \[
    V = \frac{Q}{C}
    \]
  - Let \( q \) and \( v \) be the charge and potential difference at an intermediate stage during the charging process.
    \[
    v = \frac{q}{C}
    \]
  - At this stage the work \( dW \) required to transfer an additional element of charge \( dq \) is:
    \[
    dW = vdq = \frac{q dq}{C}
    \]
  - The total work needed to increase the capacitor charge \( q \) from zero to \( Q \) is:
    \[
    W = \int_0^W dW = \frac{1}{C} \int_0^Q q dq = \frac{Q^2}{2C}
    \]
Energy Storage and Electric-field Energy

- Potential energy of a charged capacitor

  - Define the potential energy of an uncharged capacitor to be zero.

  - Then $W$ in the previous slide is equal to the potential energy $U$ of the charged capacitor

\[
U = \frac{Q^2}{2C} = \frac{1}{2} CV^2 = \frac{1}{2} QV
\]

The total work $W$ required to charge the capacitor is equal to the total charge $Q$ multiplied by the average potential difference $(1/2)V$ during the charging process.
Energy Storage and Electric-field Energy

- Electric-field energy
  - We can think of the above energy stored in the field in the region between the plates.
  - Define the energy density $u$ to be the energy per unit volume:
    \[ u = \frac{1}{2} \frac{CV^2}{Ad} = \frac{1}{2} \varepsilon_0 E^2 \]
    
    $C = \frac{\varepsilon_0 A}{d}$

This relation works for any electric field.
Example 24.9: Two ways to calculate energy stored

- Consider the spherical capacitor in Example 24.3.

\[ C = 4\pi \varepsilon_0 \frac{r_ar_b}{r_b - r_a} \]

- The energy stored in this capacitor is:

\[ U = \frac{Q^2}{2C} = \frac{Q^2}{8\pi \varepsilon_0} \frac{r_b - r_a}{r_ar_b} \]

- The electric field between two conducting sphere:

\[ E = \frac{Q}{4\pi \varepsilon_0 r^2} \]

- The electric field inside the inner sphere is zero.

- The electric field outside the inner surface of the outer sphere is zero.

\[ u = \frac{1}{2} \varepsilon_0 E^2 = \frac{1}{2} \frac{Q}{4\pi \varepsilon_0 r^2} = \frac{Q^2}{32\pi^2 \varepsilon_0 r^4} \]

\[ U = \int udV = \int_{r_a}^{r_b} \left( \frac{Q^2}{32\pi^2 \varepsilon_0 r^4} \right) 4\pi r^2 dr = \frac{Q^2}{8\pi \varepsilon_0} \int_{r_a}^{r_b} \frac{dr}{r^2} = \frac{Q^2}{8\pi \varepsilon_0} \frac{r_b - r_a}{r_ar_b} \]
Energy Storage and Electric-field Energy

Example: Stored energy

Example:
How much electric energy is stored by a solid conducting sphere of radius $R$ and total charge $Q$?

Answer:

$$U = \frac{KQ^2}{2R}$$

$$E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \rightarrow u = \frac{1}{2} \varepsilon_0 E^2$$

$$U = \int_{r}^{\pi} \frac{1}{2} \varepsilon_0 \left( \frac{1}{4\pi\varepsilon_0} \right)^2 \frac{Q^2}{r^4} dV = \frac{Q^2}{2(4\pi\varepsilon_0)R}$$

Example:
How much electric energy is stored by a solid insulating sphere of radius $R$ and total charge $Q$ uniformly distributed throughout its volume?

Answer:

$$U = \left( 1 + \frac{1}{5} \right) \frac{KQ^2}{2R} = \frac{3}{5} \frac{KQ^2}{R}$$
Dielectrics

Dielectric materials

• Experimentally it is found that when a non-conducting material (dielectrics) between the conducting plates of a capacitor, the capacitance increases for the same stored charge $Q$.
• Define the dielectric constant $\kappa$ (= $K$ in the textbook) as:

$$\kappa = \frac{C}{C_0}$$

• When the charge is constant,

$$Q = C_0V_0 = CV \rightarrow C / C_0 = V_0 / V$$

$$V = \frac{V_0}{\kappa} \rightarrow E = \frac{E_0}{\kappa}$$

<table>
<thead>
<tr>
<th>Material</th>
<th>$\kappa$</th>
<th>Material</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>vacuum</td>
<td>1</td>
<td>Mica</td>
<td>3-6</td>
</tr>
<tr>
<td>air(1 atm)</td>
<td>1.00059</td>
<td>Mylar</td>
<td>3.1</td>
</tr>
<tr>
<td>Teflon</td>
<td>2.1</td>
<td>Plexiglas</td>
<td>3.40</td>
</tr>
<tr>
<td>Polyethelene</td>
<td>2.25</td>
<td>Water</td>
<td>80.4</td>
</tr>
</tbody>
</table>
Dielectrics

- Induced charge and polarization

- Consider a two oppositely charged parallel plates with vacuum between the plates.
- Now insert a dielectric material of dielectric constant $\kappa$.  
  \[ E = \frac{E_0}{\kappa} \text{ when } Q \text{ is constant} \]
- Source of change in the electric field is redistribution of positive and negative charge within the dielectric material (net charge 0). This redistribution is called a polarization and it produces induced charge and field that partially cancels the original electric field.

\[ E_0 = \frac{\sigma}{\varepsilon_0} \quad E = \frac{\sigma - \sigma_{\text{ind}}}{\varepsilon_0} \quad E = \frac{E_0}{\kappa} \]

\[ \sigma_{\text{ind}} = \sigma \left( 1 - \frac{1}{\kappa} \right) \text{ and define the permittivity } \varepsilon = \kappa \varepsilon_0 \]

\[ E = \frac{\sigma}{\varepsilon} \quad C = \kappa C_0 = \kappa \varepsilon_0 \frac{A}{d} = \varepsilon \frac{A}{d} \quad u = \frac{1}{2} \kappa \varepsilon_0 E^2 = \frac{1}{2} \varepsilon E^2 \]
Dielectrics

- Molecular model of induced charge

**Understanding:** \[ \vec{E} = \frac{\vec{E}_0}{\kappa} \]

- The dielectric can be *polarized*.
- So at the atomic or molecular level, the positive and negative charges are slightly separated.
- Even though the molecules are electrically neutral they possess a permanent electric dipole moment.
- These dipoles are randomly oriented in the absence on an electric field.
Dielectrics

- Molecular model of induced charge (cont’d)

- If we apply a uniform electric field $E_0$ to these dipoles
- The electric field exerts a torque on each dipole that tends to align the dipole moment with the applied electric field.

The net effect is an *induced surface charge* on the dielectric that gives rise to an *induced electric field* in the dielectric.
Dielectrics

Why salt dissolves

Normally NaCl is in a rigid crystal structure, maintained by the electrostatic attraction between the Na⁺ and Cl⁻ ions.

Water has a very high dielectric constant (78). This reduces the field between the atoms, hence their attraction to each other. The crystal lattice comes apart and dissolves.
Dielectrics

- **Gauss’s law in dielectrics**

Gauss’s law:

\[ EA = \frac{(\sigma - \sigma_{\text{ind}})A}{\varepsilon_0} \]

\[ \sigma_{\text{ind}} = \sigma \left( 1 - \frac{1}{\kappa} \right) \text{ or } \sigma - \sigma_{\text{ind}} = \frac{\sigma}{\kappa} \]

\[ EA = \frac{\sigma A}{\kappa \varepsilon_0} \text{ or } \kappa EA = \frac{\sigma A}{\varepsilon_0} \]

\[ \int \kappa \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl-free}}}{\varepsilon_0} \]

enclosed free charge
An air capacitor is made by using two flat plates each with area A separated by a distance d.

(a) If the distance d is halved, how much does the capacitance change?
(b) If the area is doubled, how much does the capacitance change?
(c) For a given stored charge Q, to double the amount of energy stored how much should the distance d be changed?

Now a metal slab of thickness \( a \) (< d) and of the same area A is inserted between the two plates in parallel to the plates as shown in the figure (the slab does not touch the plates).

(d) What is the capacitance of this arrangement? (hint: serial connection)
(e) Express the capacitance as a multiple of the capacitance \( C_0 \) when the metal slab is not present.
Problem 1 Solution

(a) \[ C = \varepsilon_0 \frac{A}{d}, \text{ so } C \text{ is doubled.} \]

(b) \[ C = \varepsilon_0 \frac{A}{d}, \text{ so } C \text{ is doubled.} \]

(c) \[ C = \varepsilon_0 \frac{A}{d} \text{ and } U = \frac{Q^2}{2C}, \text{ so } U = \frac{Q^2 d}{2\varepsilon_0 A} \text{ and } d \text{ should be doubled.} \]

(d) This arrangement can be considered to be a system of two capacitors connected in series, each of which has a gap of \((d - a)/2\) between the plates.

Each of these two capacitor has the capacitance \(C = \varepsilon_0 \frac{2A}{d - a}\). Therefore

the equivalent capacitance \(C_{eq}\) is: \[ 1/ C_{eq} = 2/ C \Rightarrow C_{eq} = \varepsilon_0 \frac{A}{d - a} \]

(e) \[ C_0 = \varepsilon_0 \frac{A}{d}, \text{ therefore } C_{eq} = \frac{d}{d - a} C_0 \]
Problem 2

In this problem you try to measure dielectric constant of a material. First a parallel-plate capacitor with only air between the plates is charged by connecting it to a battery. The capacitor is then disconnected from the battery without any of the charge leaving the plates.

(a) Express the capacitance $C_0$ in terms of the potential difference $V_0$ between the plates and the charge $Q$ if air is between the plates.

(b) Express the dielectric constant $\kappa$ in terms of the capacitance $C_0$ (air gap) and the capacitance $C$ with material of the dielectric constant $\kappa$.

(c) Using the results of (a) and (b), express the ratio of the potential difference $V/V_0$ if $Q$ is the same, where $V$ is the potential difference between the plates and a dielectric material dielectric constant is $\kappa$ fills the space between them.

(d) A voltmeter reads 45.0 V when placed across the capacitor. When dielectric material is inserted completely filling the space, the voltmeter reads 11.5 V. Find the dielectric constant of this material.

(e) What is the voltmeter read if the dielectric is now pulled partway out so that it fills only one-third of the space between the plates? (Use the formula for the parallel connection of two capacitors.)
Problem 2

(a) \( C_0 = Q / V_0 \)

(b) \( \kappa = C / C_0 \)

(c) \( V_0 / V = C / C_0 = \kappa \rightarrow V / V_0 = 1 / \kappa \)

(d) From (c) \( \kappa = V_0 / V = 45.0 / 11.5 = 3.91 \)

(e) In the new configuration the equivalent capacitor is \( C_{eq} = C_{1/3} + C_{0,2/3} \) where \( C_{1/3} \) is the contribution from the part that has the dielectric material and \( C_{0,2/3} \) is the part that has air gap. \( C_{1/3} = (1/3)C \) and \( C_{0,2/3} = (2/3)C_0 \) because the capacitance is proportional to the area.

\[
C_{eq} = C_{1/3} + C_{0,2/3} = (1/3)C + (2/3)C_0 = C_0[(1/3)\kappa + (2/3)]
\]

Using the results from (c)

\[
V_0 / V = C_{eq} / C_0 = [(1/3)\kappa + (2/3)] \rightarrow \\
V = V_0 / [(1/3)\kappa + (2/3)] = (45.0 \text{ V})\left(\frac{3}{5.91}\right) = 22.8 \text{ V}
\]